TOPOLOGICAL GROUPS WITH EQUAL LEFT AND RIGHT UNIFORMITIES

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If (G, σ) is a topological group with topology σ, there is a natural topology τ on G such that $\tau \supset \sigma$, (G, τ) is a topological group with equal left and right uniformities and τ is the smallest such topology. The topology τ can be obtained as follows. If $\mathcal{V}$ is a neighborhood base at the identity in $(G, \sigma) = G$, then $\left\{ \bigcap_{t \in G} tVt^{-1} \mid V \in \mathcal{V} \right\}$ is a neighborhood base at the identity in $(G, \tau) = G^*$. Here we are interested in determining the nature of G^* given G and also in obtaining results on groups with equal left and right uniformities. For results on such groups we point to Braconnier [1], in which it is shown that a locally compact group with equal left and right uniformities has a two sided invariant Haar measure, and to N. Rickert's unpublished theorems on the structure of locally compact groups with equal uniformities. In contrast, T. S. Wu [10] has shown that left almost periodic functions (in the sense of von Neumann) on locally compact groups need not be right almost periodic. Of course, the groups here must have distinct left and right uniformities.

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These groups were discovered at an intermediate stage in the construction of a locally compact group G for which G^* is not locally compact. This example will be described later.

One of our principal results (Theorem 1) is that G^* is a Lie group if G is a connected Lie group. Since this depends only on the fact that G^* is locally compact, we have attempted to determine more precisely when this occurs. As mentioned above, there are locally compact groups G for which G^* is not locally compact; however, if G is locally compact and G/G_o is compact, where G_o is the identity component, then G^* is locally compact. It is also shown that if a connected locally compact group G with equal left and right uniformities contains a one parameter subgroup $\varphi(R)$ as a normal subgroup, then $\varphi(R)$ is contained in the center of G. The hypothesis on the uniformities is not necessary if $G/\overline{\varphi(R)}$ is compact. These are Corollaries 1 and 2 of Theorem 3.

As a preliminary we obtain some lemmas on the topology of a subgroup $H \subset G$ relative to G^*.

<u>Lemma 1</u>. If K is a compact normal subgroup of the connected group G, then the topologies of K relative to G and G^* are equal.

Proof. Let $A(K)$ be the group of all continuous automorphisms of K and $I(K)$ the group of inner automorphisms of K. Define $\Delta: G \to A(K)$ as follows: $\Delta(g)(k) = gkg^{-1}$. We note that Δ is continuous when $A(K)$ has the compact-open = uniform topology. Thus $\Delta(G)$ is a connected subgroup of $A(K)$. It follows that $\Delta(G) = I(K)$ since $A(K)/I(K)$ is totally disconnected by Theorem 1 of [6]. Let U be a neighborhood of the identity in G. By compactness of K we pick a neighborhood V such that $V \subset \bigcap_{k \in K} kUk^{-1}$. We have,

$$K \cap \left(\bigcap_{t \in G} tUt^{-1}\right) = \bigcap_{t \in G} t(U \cap K)t^{-1} = \bigcap_{k \in K} k(U \cap K)k^{-1} \supset V \cap K.$$

This proves the lemma.

Lemma 2. If H is a subgroup of G such that the coset space G/H is compact, then the sets $\bigcap_{h \in H} hVh^{-1}$, where V is a neighborhood of e in G, form a neighborhood base at e in G^*.

Proof. Let V be a neighborhood of e in G and let W be a symmetric neighborhood of e such that $W^3 \subset V$. Since G/H is compact, $G = \bigcup_{i=1}^{n} Wg_iH$ for some sequence $g_1, \ldots, g_n$ in G. Now

$$V_i = \bigcap_{t \in Wg_iH} t^{-1}Vt \supset \bigcap_{h \in H} h^{-1}g_i^{-1}Wg_ih \text{ and}$$

$$\bigcap_{t \in G} t^{-1}Vt = \bigcap_{i=1}^{n} V_i \supset \bigcap_{h \in H} h^{-1}\left(\bigcap_{i=1}^{n} g_i^{-1}Wg_i\right)h.$$

This proves the lemma.

Corollary 1. If in addition to the hypothesis of Lemma 2, H has equal left and right uniformities, then the topologies of H relative to G and G* are equal.

Corollary 2. If G is a locally compact group and G/H is compact, where H is a normal subgroup with equal left and right uniformities, then G* is locally compact.

Proof. The closure C of H is locally compact and has equal left and right uniformities. It follows from Corollary 1 that C is locally compact in G*. Since G/C is compact we have $(G/C)^* = G/C$. But $(G/C)^* = G^*/C$ for any group G and normal subgroup C. Now by a theorem of Gleason (1.10 [3]) G* is locally compact.

Theorem 1. If G is a connected Lie group, then G* is a Lie group.

Proof. It is sufficient to prove that G* is locally compact. To do this we use 1.12, page 100 [4]. Let M be the tangent space of G at e. For $g \in G$ let φ_g be the inner automorphism determined by g and denote by $d\varphi_g$ the differential of φ_g. Now $d\varphi_g$ is an isomorphism of M and $\exp d\varphi_g = \varphi_g \exp$, where exp is the exponential map of M to G. Now let W be a neighborhood of the origin in M such that $\exp|W$ is a homeomorphism. There are neighborhoods U and V of e in G such that $V = \exp Z$, where Z is convex, U is symmetric and $gVg^{-1} \subset \exp W$ for $g \in U$. Since $g(\exp Z)g^{-1} \subset \exp W$, $d\varphi_g(Z) \subset W$ for $g \in U$. We obtain a sequence of sets $\{V_n\}$ as follows:

$$V_1 = \bigcap_{g \in U} gVg^{-1}$$

$$V_n = \bigcap_{g \in U} gV_{n-1}g^{-1} = \bigcap_{g \in U^n} gVg^{-1}$$

Let $F(X) = \bigcap_{g \in U} d\varphi_g(X)$. We show

(α). $\exp F^n(Z) = V_n$

First $\exp F(Z) = \exp \bigcap_{g \in U} d\varphi_g(Z) = \bigcap_{g \in U} \exp d\varphi_g(Z)$ since $\exp$ is 1-1 on $d\varphi_g(Z)$. Thus, $\exp F(Z) = \bigcap_{g \in U} \varphi_g(\exp Z) = V_1$. This proves ($\alpha$) for $n = 1$ and since $V_1 \subset V$ we have also shown that $F(Z) \subset Z$ and hence $F^n(Z) \subset Z$. It follows that $\exp$ is 1-1 on $d\varphi_g(F^n(Z))$. Now suppose that (α) holds for the positive integer n. Then $\exp F^{n+1}(Z) =$ $\exp \bigcap_{g \in U} d\varphi_g(F^n(Z)) = \bigcap_{g \in U} \varphi_g(V_n) = V_{n+1}$ and (α) is proved.

Since Z is convex $F^n(Z)$ and hence $\bigcap_{n=1}^{\infty} F^n(Z)$ is convex. Thus $\exp \bigcap F^n(Z) = \bigcap \exp F^n(Z) = \bigcap V_n$ is locally Euclidean. But $\bigcap_n V_n = \bigcap_n \bigcap_{g \in U^n} gVg^{-1} = \bigcap_{g \in G} gVg^{-1}$. This proves the theorem.

Lemma 3. If G is a connected locally compact group, then G^* is locally compact.

Proof. Let K be a compact normal subgroup of G such that G/K is a Lie group. By Theorem 1 $(G/K)^*$ is locally compact. Thus G^*/K is locally compact and K is compact in G^* by Lemma 1. Again using 1.10 [3] we have G^* locally compact.

Theorem 2. If G is a locally compact group such that G/G_0 is compact, then G^* is locally compact.

Proof. By Lemma 2 the topology of G_0 relative to G^* is obtained by taking the sets $\bigcap_{g \in G_0} gVg^{-1} \cap G_0$ where V is a neighborhood of e in G. It follows from Lemma 3 that G_0 is locally compact relative to G^*. The theorem follows since G^*/G_0 is compact.

The following is an example of a locally compact group G such that G^* is not locally compact.

Example. Let $A_i = Z_2$, the discrete two element group, for each integer i. Let $H_0 = \prod_i A_i$ and let Z be discrete infinite cyclic with generator z. Define an operation in $G_0 = Z \times H_0$ as follows, $(z^n, x)(z^m, y) = (z^{n+m}, x_{-m}y)$ where the i^{th} coördinate of x_{-m} is the $(i-m)^{th}$ coordinate of x. Thus G_0 is a semidirect product [5]. It is easy to see that G_o is a locally compact group with this operation and the usual product topology. Let $G_i = G_o$ and $H_i = \{e\} \times H_o$ for $i = 1,2,\ldots$. Let $H = \prod H_i$ and $G = \prod G_i$ with coördinatewise operation, the usual product topology on H and H open in G. Thus, G is a locally compact group and H is a compact open subgroup of G. Let V_i be a proper neighborhood of e in H_i. If $U_n = V_1 \times \ldots \times V_n \times (\prod_{i>n} H_i)$, then $\bigcap_{t \in G} tU_nt^{-1} = \{e\} \times \ldots \times \{e\} \times (\prod_{i>n} H_i)$. Since $\{U_n\}$ is a neighborhood base at the identity of G, it follows that G* is not locally compact.

A more general construction than that above is obtained by letting F be a compact infinite group which admits an expansive automorphism φ. Let H be the discrete group generated by φ. Now replace G_o by the semidirect product $F \circledS H$ in the above example. The existence of expansive automorphisms on compact groups has been established by R. F. Williams [9], T. S. Wu [11], M. Eisenberg [2] and F. Reddy [8]. At the end of this paper we establish the existence of expansive homeomorphisms on compact groups which are not automorphisms.

By a one parameter subgroup in a group G we mean the image $\varphi(R)$ of the reals under a continuous homomorphism $\varphi: R \to G$.

Theorem 3. If G is a connected locally compact group and $\varphi(R)$ is a one parameter subgroup of G* which is normal, then $\varphi(R)$ is contained in the center of G.

Proof. By Lemma 3 G* is locally compact. Thus φ is a topological isomorphism or $\overline{\varphi(R)}$ is compact in G* (page 102 [7]). In case $\overline{\varphi(R)}$ is compact in G* it is also compact in G and hence contained in the center of G (26.10 [5]). Now suppose that φ is an isomorphism. Pick a neighborhood V of e in G such that $\bigcap_{g \in G} gVg^{-1} \cap \varphi(R) = (-\epsilon, \epsilon)$. Let α be in $(0, \epsilon)$ and define a sequence in $(-\epsilon, \epsilon)$ as follows, $\alpha_1 = g\alpha g^{-1}$, $\alpha_n = g\alpha_{n-1}g^{-1}$ where g is any element of G. The sequence $\{\alpha_n\}$ is monotone. Let $\beta = \lim \alpha_n$. It follows that $g\beta g^{-1} = \beta$. We can assume that β is non zero since, if $\{\alpha_n\}$ is decreasing, then on replacing g by g^{-1} we get an increasing sequence and its limit will also commute with g. It is easy to show that, if g commutes with a non-zero real number β then it commutes with every rational multiple of β hence with every real number. Thus, $\varphi(R)$ is contained in the center of G.

As an immediate consequence of Theorem 3 we have.

Corollary 1. If G is a connected locally compact group with equal left and right uniformities which contains a one parameter group as a normal subgroup, then this subgroup is in the center of G.

Corollary 2. If G is a connected locally compact group which contains a one parameter normal subgroup $\varphi(R)$ such that $G/\overline{\varphi(R)}$ is compact, then $\varphi(R)$ is contained in the center of G.

Proof. If $\varphi(R)$ is compact, then the conclusion follows from 26.10 [5]. Otherwise φ is an isomorphism (page 102 [7]) and by Corollary 1 of Lemma 2 $\varphi(R)$ is a one parameter subgroup of G*. The corollary now follows from the theorem.

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